Control to facet by piecewise-affine output feedback

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Abstract The control-to-facet problem plays an important role in the design of feedback controllers for piecewise-affine hybrid systems on polytopes. In the literature, necessary and sufficient conditions for solvability by static state feedback exist. In this paper, we extend these results to the case of continuous piecewise-affine output feedback. For the construction of a controller, a triangulation of the output polytope is made, that satisfies additional conditions, to guarantee compatibility with the induced subdivision of the state polytope. In the state feedback case, the use of this special type of triangulations was not required.

Key words affine system on polytope, output feedback, control to facet, control of hybrid systems, triangulation.

1 Introduction and motivation

In the past fifteen years, the study of hybrid systems has become a very active research area. There are several reasons for this rapidly growing interest. Nowadays many engineering systems are controlled by computers, which creates an interaction between the continuous dynamics of a physical system, and the discrete dynamics of a computer. Furthermore, the dynamics of control systems often contain discontinuities, or become hybrid after modelling. Examples of hybrid systems are abundantly

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available, ranging from the modelling of car engines to (air) traffic control and robot motion planning.

One particular class of hybrid systems, the class of *piecewise-affine hybrid systems*, has received a large amount of attention. A piecewise-affine hybrid system consists of a discrete automaton with a continuous-time affine system on a polyhedral set at each discrete mode, and a switching mechanism between discrete and continuous dynamics. This class of systems was introduced by Sontag in [8] and [9], and has become popular because they are appropriate for the modelling of real-world systems, and because their mathematical structure allows for useful theoretical results.

Several approaches exist for controlling piecewise-affine hybrid systems. A popular approach, developed by Morari, Bemporad et al. (see e.g. [1, 2]) is based on the time discretization of the system, and is geared to performance optimization by computational methods. A completely different methodology is the so-called control-to-facet approach (see [3, 4, 5, 6]). The idea of this approach is to apply continuous state feedback at each discrete mode of the hybrid system in order to influence the switching behavior of the underlying discrete automaton. Assuming that the switching depends on the facet through which a polytope is left, a central question in this method is how an affine system on a polytope can be steered to a particular (set of) facet(s), without leaving the state polytope before reaching this so-called exit facet. The solution to this control problem becomes a building block for the control of hybrid systems. In [6] it has been combined with a backward recursion algorithm on the discrete dynamics in order to achieve some a priori given control objectives.

In this paper, we focus on one discrete mode of a hybrid system, and consider the control-to-facet problem on a polytope. In [4] and [6] several problems of this type have been solved by *state feedback*. In this paper we generalize these results to the case of partial observations, and consider the control-to-facet problem with *static output feedback*. In both cases, the construction of a control law involves triangulating the state/output polytope, and solving a system of linear inequalities at each vertex of the triangulation. The main differences between the output feedback and the state feedback constructions are that for output feedback, the triangulation of the output polytope has to be carried out before the solvability of the system of linear inequalities is checked, and that the triangulation of the output polytope has to satisfy additional conditions, while in the state-feedback case any triangulation of the state polytope is allowed.

2 Problem description

Let $X \subset \mathbb{R}^n$ be a full-dimensional polytope. Denote by $\mathcal{V}(X)$ the set of vertices of X, and by $\mathcal{F}(X)$ the set of facets of X. If $\hat{x}$ is a point on the boundary of X, then we denote by $\mathcal{F}(\hat{x}, X)$ the set of all facets of X which contain the point $\hat{x}$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and consider the autonomous system $\dot{x} = f(x)$ on the polytope X. Let F be a facet of X with outward normal vector n_F. Then the *exit set* $\mathrm{Ex}(F)$ of F is defined as the set of points of F through which a state trajectory $x(t)$ can leave X, i.e.

$$\mathrm{Ex}(F) = \mathrm{cl}\left\{x \in F \mid n_F^T f(x) > 0\right\}.$$

The facet F is *blocked* if $\mathrm{Ex}(F) = \varnothing$.

Definition 1. *Let $x(t, x_0)$ be the trajectory of the autonomous system $\dot{x} = f(x)$ on polytope X with initial state $x_0 \in X$. This trajectory is said to* cross *facet F of X at time $T \geq 0$ if*

(i) $\forall t \in [0, T] :\ x(t, x_0) \in X$*;*

(ii) $\exists \varepsilon > 0,\ \forall t \in (T, T + \varepsilon) :\ x(t, x_0) \notin X$*;*

(iii) $x(T, x_0)$ *is an element of the exit set of F.*

In this paper we study an affine control system Σ on a full-dimensional polytope X in $\mathbb{R}^n$, given by

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)) = Ax(t) + Bu(t) + a, & x(0) = x_0; \\ y(t) = \quad h(x(t)) \quad = Cx(t) + c. & \end{cases} \tag{1}$$

Here $x(t) \in X$ denotes the state, $u(t) \in U$ the input, with U a polytope in $\mathbb{R}^m$, and $y(t) \in Y$ the output, with $Y = CX + c$ a polytope in $\mathbb{R}^p$. In particular, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $c \in \mathbb{R}^p$. Without loss of generality we assume that C is of full row rank; in particular $p \leq n$. The dynamics described in (1) remain valid as long as the state x remains in the state-polytope X. Let $\mathcal{E} \subset \mathcal{F}(X)$ denote a set of *admissible exit facets*. We want to solve the problem of steering the state $x(t)$ to an admissible exit facet, using static *output* feedback. In a hybrid systems context this corresponds to the the disabling of unfavorable discrete transitions from a given discrete location.

Problem 2. *(Control-to-facet) Find a continuous piecewise-affine map $g : Y \longrightarrow U$ such that the static output feedback law $u(t) = g(y(t))$ guarantees that all state trajectories of the closed-loop system $\dot{x} = Ax + Bg(Cx + c) + a$, with initial state $x(0) = x_0 \in X$, can only leave X by crossing an admissible exit facet $F \in \mathcal{E}$. In other words, all non-admissible exit facets $F \in \mathcal{F}(X) \backslash \mathcal{E}$ are blocked w.r.t. the closed-loop dynamics.*

For our solution of the output feedback Problem 2, we use a similar approach to that used in [4, 6]. We first construct a triangulation of the output polytope which satisfies certain *compatibility* conditions with the state polytope and output map. We then set up a system of linear inequalities for the input at each vertex of the triangulation. If these inequalities have a solution, we construct the output feedback control law by interpolating the input at the vertices of the triangulation.

The problem with partial observations is more difficult than that with complete observations because one observed output $y \in Y$ corresponds to a *set* of states

$X_y = \{x \in X \mid Cx + c = y\}$. Furthermore, the set X_y may intersect several facets of X, so the corresponding input $u = g(y)$ has to satisfy constraints with respect to all these facets at the same time.

3 Triangulation of the output polytope

In order to handle all constraints on the input corresponding to a given output, we first make a triangulation of the output polytope in such a way that the inverse images of all outputs in one particular simplex of the triangulation intersect the same facets of the state polytope X.

Definition 3. *Let X be a full-dimensional polytope in $\mathbb{R}^n$, and let $h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a surjective affine function i.e. $h(x) = Cx + c$ where C has full row rank. Let $Y = h(X)$, so Y is a full-dimensional polytope in $\mathbb{R}^p$. A triangulation T of Y is a set of full-dimensional simplices $S_1, \dots, S_L$ in $\mathbb{R}^p$ with the following properties*

(i) $\displaystyle\bigcup_{i=1}^{L} S_i = Y$,

(ii) For all $i, j \in \{1, \dots, L\}$ with $i \neq j$ the intersection $S_i \cap S_j$ is either empty, or a common face of S_i and S_j.

The vertex set *of a triangulation consists of the union of all vertices of the simplices in the triangulation: $\mathcal{V}(T) := \bigcup_{i=1}^{L} \mathcal{V}(S_i)$.*

A triangulation T of Y is (X, h)-compatible *if additionally*

(iii) $\forall S \in T : \ \mathcal{V}(h^{-1}(S) \cap X) \subset h^{-1}(\mathcal{V}(S))$,

where $h^{-1}(W) := \{x \in \mathbb{R}^n \mid h(x) \in W\}$ denotes the inverse image of a set $W \subset \mathbb{R}^p$ under the affine map h.

In other words, a triangulation T of Y is (X, h)-compatible, if for every simplex $S \in T$, the vertices of the polytope $h^{-1}(S) \cap X$ are mapped to vertices of S by the mapping h. Hence $h^{-1}(S) \cap X$ is the convex hull of $h^{-1}(\mathcal{V}(S)) \cap X$.

It is easy to prove that the images of all vertices of the polytope X under the affine mapping h become elements of the vertex set of an (X, h)-compatible triangulation:

Lemma 4. *Let T be an (X, h)-compatible triangulation of polytope $Y = h(X)$. Then*

$$h(\mathcal{V}(X)) \subset \mathcal{V}(T). \tag{2}$$

In general, inclusion (2) is strict, and the vertex set $\mathcal{V}(T)$ of an (X, h)-compatible triangulation will contain more points than just the images of the vertices of X under the mapping h.

In order to generate a compatible triangulation, we start with the *chamber complex* of (X, h) as defined in [7].

Definition 5. *Let X be a polytope and h a surjective affine map. The chamber complex Γ of (X,h) is the polyhedral subdivision of $Y = h(X)$ with faces $\{\sigma(y) \mid y \in Y\}$ where*

$$\sigma(y) = \bigcap \{h(F) \mid F \text{ is a face of } X \text{ and } y \in h(F)\}.$$

The following result shows that any simplicial refinement of the chamber complex is a compatible triangulation:

Theorem 6. *Let T be a triangulation of Y which refines the chamber complex Γ of (X,h). Then T is (X,h)-compatible.*

The proof follows from [7, Proposition 2.4]. Hence we can construct an (X,h)-compatible triangulation of Y by first computing the chamber complex of (X,h), and then taking a suitable triangulation of each chamber.

4 Solvability conditions and construction of a piecewise-affine output feedback

If T is an (X,h)-compatible triangulation of the output polytope Y, and for every vertex $w \in \mathcal{V}(T)$ a corresponding input $u_w \in U$ is fixed, then an admissible piecewise affine output feedback is easily constructed. Any $y \in Y$ is contained in at least one $S_y \in T$, and can be written (in a unique way) as a convex combination of the vertices of S_y:

$$y = \sum_{w \in \mathcal{V}(S_y)} \lambda_{y,w} w,$$

with $\lambda_{y,w} \in [0,1]$, $w \in \mathcal{V}(S_y)$, and $\sum_{w \in \mathcal{V}(S_y)} \lambda_{y,w} = 1$. Next we define the output feedback $g : Y \longrightarrow U$ by

$$g(y) = \sum_{w \in \mathcal{V}(S_y)} \lambda_{y,w} u_w. \tag{3}$$

Then g is a continuous function, and affine on every simplex of T.

Next, we recall that $h(\mathcal{V}(X)) \subset \mathcal{V}(T)$ (Lemma 4). In every vertex $w \in \mathcal{V}(T)$ we want to find an input $u_w \in U$ such that in all points in $h^{-1}(w) \cap \partial X$ the vector field is pointing in the right direction. In particular, in all vertices of non-admissible exit facets of X, the closed-loop vector field should point into the polytope X. If it is possible to realize these conditions at the vertices, then they also remain valid on the facet, because facets are convex sets, the closed-loop dynamics is piecewise-affine, and the triangulation T is (X,h)-compatible. These observations lead to the main result of this paper:

Theorem 7. *Consider an affine system (1) on a full-dimensional polytope X in $\mathbb{R}^n$. Let $\mathcal{E} \subset \mathcal{F}(X)$ be a set of admissible exit facets. Let T be an (X,h)-compatible triangulation of the output polytope Y, with $h(x) = Cx + c$ the affine output map. Then Problem 2 is solvable by a continuous piecewise-affine output feedback $g : Y \longrightarrow U$ if and only if*

$$\forall w \in \mathcal{V}(T),\ \exists u_w \in U,\ \forall v \in \mathcal{V}(h^{-1}(w)\cap X),\ \forall F \in \mathcal{F}(v,X)\backslash\mathcal{E}: \quad n_F^T(Av + Bu_w + a) \le 0, \tag{4}$$

where n_F denotes the unit normal vector of facet F, pointing out of polytope X.

Proof. Necessity is straightforward, and is shown in a similar way as in [4] and [6].

To prove sufficiency, assume that in all vertices $w \in \mathcal{V}(T)$ an input u_w is chosen such that (4) holds. We will prove that the corresponding piecewise-affine feedback law $u = g(y)$, as described in (3), solves Problem 2, by verifying that

$$\forall F \in \mathcal{F}(X)\backslash\mathcal{E},\ \forall x \in F:\ n_F^T(Ax + Bg(h(x)) + a) \le 0.$$

Let $\hat{F} \in \mathcal{F}(X)\backslash\mathcal{E}$, and let $\hat{x} \in \hat{F}$. Define $\hat{y} := h(\hat{x})$. There is at least one simplex $S \in T$ such that $\hat{y} \in S$. Furthermore, $\hat{y}$ can uniquely be represented as a convex combination of vertices of S, i.e. there exist $w_1,\ldots,w_k \in \mathcal{V}(S)$ and *unique* $\lambda_1,\ldots,\lambda_k \in (0,1]$ such that $\sum_{i=1}^k \lambda_i w_i = \hat{y}$, and $\sum_{i=1}^k \lambda_i = 1$.

Clearly, $\hat{x} \in h^{-1}(S)\cap X$, and since triangulation T is (X,h)-compatible, we know from Condition *(iii)* of Definition 3 that

$$\forall v \in \mathcal{V}(h^{-1}(S)\cap X):\ h(v) \in \mathcal{V}(S).$$

Let $\mathcal{V}(S) = \{w_1,\ldots,w_k,w_{k+1},\ldots,w_{p+1}\}$. Then there exist

$$\begin{aligned} v_{1,1},\ldots,v_{1,\ell_1} &\in \mathcal{V}(h^{-1}(w_1)\cap X),\\ v_{2,1},\ldots,v_{2,\ell_2} &\in \mathcal{V}(h^{-1}(w_2)\cap X),\\ &\vdots\\ v_{p+1,1},\ldots,v_{p+1,\ell_{p+1}} &\in \mathcal{V}(h^{-1}(w_{p+1})\cap X), \end{aligned}$$

and scalars $\rho_{1,1},\ldots,\rho_{1,\ell_1},\rho_{2,1},\ldots,\rho_{2,\ell_2},\ldots,\rho_{p+1,1},\ldots,\rho_{p+1,\ell_{p+1}} \in [0,1]$ such that

1. $\sum_{i=1}^{p+1}\sum_{j=1}^{\ell_i} \rho_{i,j} v_{i,j} = \hat{x}$,
2. $\sum_{i=1}^{p+1}\sum_{j=1}^{\ell_i} \rho_{i,j} = 1$.

For $i = 1,\ldots,p+1$ we define $\mu_i := \sum_{j=1}^{\ell_i} \rho_{i,j}$. Then $\mu_i \in [0,1]$ and $\sum_{i=1}^{p+1}\mu_i = 1$. Subsequently we define for $i = 1,\ldots,p+1$ and $j = 1,\ldots,\ell_i$

$$\alpha_{i,j} = \begin{cases} 0 & \text{if} \quad \mu_i = 0,\\ \frac{\rho_{i,j}}{\mu_i} & \text{if} \quad \mu_i \neq 0. \end{cases}$$

Then

$$\hat{x} = \sum_{i=1}^{p+1}\sum_{j=1}^{\ell_i} \rho_{i,j} v_{i,j} = \sum_{i=1,\mu_i\neq 0}^{p+1} \mu_i \sum_{j=1}^{\ell_i} \frac{\rho_{i,j}}{\mu_i} v_{i,j} = \sum_{i=1}^{p+1} \mu_i \sum_{j=1}^{\ell_i} \alpha_{i,j} v_{i,j}.$$

For $i = 1, \dots, p+1$, we define $z_i = \sum_{j=1}^{\ell_i} \alpha_{i,j} v_{i,j}$. Then $\mu_i = 0$ implies $z_i = 0$, and $\mu_i \neq 0$ implies that $z_i \in \mathrm{Conv}(\{v_{i,1}, \dots, v_{i,\ell_i}\})$, where Conv denotes the convex hull. Hence, if $\mu_i \neq 0$ then $z_i \in \mathrm{Conv}(\mathcal{V}(h^{-1}(w_i) \cap X)) = h^{-1}(w_i) \cap X$, so in particular $h(z_i) = w_i$. Since the output map h is affine, it follows that

$$\hat{y} = h(\hat{x}) = h(\sum_{i=1}^{p+1} \mu_i z_i) = \sum_{i=1, \mu_i \neq 0}^{p+1} \mu_i h(z_i) = \sum_{i=1, \mu_i \neq 0}^{p+1} \mu_i w_i = \sum_{i=1}^{p+1} \mu_i w_i.$$

On the other hand, $\hat{y} = \sum_{i=1}^{k} \lambda_i w_i$ was the *unique* representation of $\hat{y}$ as convex combination of $w_1, \dots, w_{p+1}$. It follows that $\mu_i = 0$ for all $i = k+1, \dots, p+1$, and $\mu_i = \lambda_i \neq 0$ for $i = 1, \dots, k$. We conclude that

$$\hat{x} = \sum_{i=1}^{k} \lambda_i z_i,$$

with $z_i \in h^{-1}(w_i) \cap X$, $(i = 1, \dots, k)$.

From the assumption that $\hat{x} \in \hat{F}$, it also follows that $z_i \in \hat{F}$ for all $i = 1, \dots, k$. Indeed, since $\hat{F}$ is a facet of X, there exists $\alpha_{\hat{F}} \in \mathbb{R}$ such that

$$\begin{aligned} \forall x \in X \backslash \hat{F} :\quad & n_{\hat{F}}^T x < \alpha_{\hat{F}}, \\ \forall x \in \hat{F} :\quad & n_{\hat{F}}^T x = \alpha_{\hat{F}}. \end{aligned}$$

Since $\hat{x} \in \hat{F}$, we know that $n_{\hat{F}}^T \hat{x} = \alpha_{\hat{F}}$. If there would exist an $i \in \{1, \dots, k\}$ such that $z_i \notin \hat{F}$, then $n_{\hat{F}}^T z_i < \alpha_{\hat{F}}$, and it would follow that

$$n_{\hat{F}}^T \hat{x} = \sum_{i=1}^{k} \lambda_i n_{\hat{F}}^T z_i < \sum_{i=1}^{k} \lambda_i \alpha_{\hat{F}} = \alpha_{\hat{F}},$$

which contradicts the fact that $n_{\hat{F}}^T \hat{x} = \alpha_{\hat{F}}$.

At this point it is proved that for all $i = 1, \dots, k$, the constructed z_i is an element of the polytope $h^{-1}(w_i) \cap \hat{F}$. Since $\mathcal{V}(h^{-1}(w_i) \cap \hat{F}) \subset \mathcal{V}(h^{-1}(w_i) \cap X)$, and $\hat{F} \in \mathcal{F}(v, X) \backslash \mathcal{E}$ for all $v \in \mathcal{V}(h^{-1}(w_i) \cap \hat{F})$, condition (4) states that

$$\forall i \in \{1, \dots, k\} \forall v \in \mathcal{V}(h^{-1}(w_i) \cap \hat{F}) : n_{\hat{F}}^T (Av + Bu_i + a) \leq 0. \tag{5}$$

Furthermore, each z_i is a convex combination of vertices in $\mathcal{V}(h^{-1}(w_i) \cap \hat{F})$, hence (5) implies that

$$n_{\hat{F}}^T (Az_i + Bu_{w_i} + a) \leq 0 \text{ for all } i = 1, \dots, k.$$

So the vector field of the closed-loop system in $\hat{x}$ satisfies

$$\begin{aligned} n_{\hat{F}}^T (A\hat{x} + Bg(h(\hat{x})) + a) &= n_{\hat{F}}^T \left(A \sum_{i=1}^{k} \lambda_i z_i + B \sum_{i=1}^{k} \lambda_i u_{w_i} + a \right) \\ &= \sum_{i=1}^{k} \lambda_i n_{\hat{F}}^T (Az_i + Bu_{w_i} + a) \leq 0. \end{aligned}$$

This completes the proof. □

Remark 8. *The proof of Theorem 7 is* completely constructive*. The result does not only describe necessary and sufficient conditions for solvability of Problem 2, but also yields a continuous piecewise-affine output feedback law that realizes this solution, provided that such a control law exists i.e. provided that there exists a solution that satisfies all linear inequalities in (4) simultaneously.*

Remark 9. *In Problem 2 it is only required to find an output feedback such that all non-admissible exit facets of the closed-loop system are blocked. This implies that either the state will leave the state polytope in finite time by crossing an admissible exit facet, or the state will remain inside the polytope forever. In a hybrid systems context this second case may be interpreted as deadlock in a hybrid mode. To prevent this from happening, one often wants to guarantee that all trajectories leave the state polytope in finite time. A sufficient condition for this requirement is that there exists a direction in $\mathbb{R}^n$ such that in all points of the state polytope the closed-loop dynamics has a strictly positive velocity in that direction. Because of the convexity of the problem, it suffices to check this condition at those points on the boundary of the state polytope that are mapped to vertices of the triangulation of the output polytope. The condition may be stated as a (bi)linear inequality on the inputs at these vertices similar to (4).*

5 An example

Consider the system

$$\dot{x} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} u + \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}; \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x.$$

on the state polytope $X = [0,1]^3$ with input set $U = [-2,2]$. Clearly, the output polytope Y is $[0,1]^2$. We want to solve Problem 2 with the facets $x_1 = 1$ and $x_2 = 1$ as admissible exit facets.

Let $w_1 = (0,0)$, $w_2 = (1,0)$, $w_3 = (1,1)$, and $w_4 = (0,1)$. Then there are two triangulations possible, one based on the diagonal from w_1 to w_3 and the other based on the diagonal from w_2 to w_4. Both triangulations are (X,h)-compatible. Every vertex w of Y corresponds to two vertices of X, with z-coordinate 0 and 1, respectively. The input u_w has to be chosen in such a way that in both these vertices the inequalities of (4) are satisfied. Straightforward computations show that the input choice $u_{w_1} = 2/3$, $u_{w_2} = u_{w_3} = u_{w_4} = 1$ satisfies all constraints. So, a continuous piecewise-affine output feedback law is given by

$$u = g(y) = \begin{cases} 1 & \text{if} \quad y_1 + y_2 \geq 1, y_1 \leq 1, y_2 \leq 1; \\ \frac{y_1+y_2+2}{3} & \text{if} \quad y_1 + y_2 < 1, y_1 \geq 0, y_2 \geq 0. \end{cases} \tag{6}$$

It is easily verified that in all vertices of X the closed-loop vector field $\dot{x} = Ax +$

$Bg(Cx) + a$ has a strictly positive component in the direction of $n = (1, 1, 0)^T$. Therefore all trajectories of the closed-loop system will leave X in finite time.

Remark 10. *In the example described above the control-to-facet problem using output feedback is also solvable if only the output $z(t) = x_1(t) + x_2(t)$ is measured. In that case, the output feedback law (6) becomes*

$$u = g_2(z) = \begin{cases} 1 & \textit{if} \quad 1 < z \leq 2; \\ \frac{z+2}{3} & \textit{if} \quad 0 \leq z \leq 1. \end{cases}$$

6 Conclusion

In this paper we extended the results on control-to-facet by state feedback derived in [4] and [6] to the case of static output feedback. We introduced the notion of (X, h)-compatible triangulations, and outlined a construction of such a triangulation. We showed that necessary and sufficient conditions for the solution of the control-to-facet problem can be stated as linear inequalities on the inputs at the vertices of the triangulation. Provided that these conditions are satisfied, an explicit construction of a static output feedback was obtained.

Bibliography

[1] A. Bemporad, G. Ferrari-Trecate, and M. Morari, *Observability and controllability of piecewise affine and hybrid systems*, in Proceedings 41st IEEE Conf. on Decision and Control, New York, IEEE Press, 1999, pp. 3966–3971.

[2] A. Bemporad and M. Morari, *Control of systems integrating logic, dynamics, and constraints*, Automatica, 35 (1999), pp. 407–427.

[3] L.C.G.J.M. Habets and J.H. van Schuppen, *Control of piecewise-linear hybrid systems on simplices and rectangles*, in Hybrid Systems: Computation and Control, M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli, eds., Lecture Notes in Computer Science, Vol. 2034, Springer-Verlag, Berlin, 2001, pp. 261–274.

[4] L.C.G.J.M. Habets and J.H. van Schuppen, *A control problem for affine dynamical systems on a full-dimensional polytope*, Automatica, 40 (2004), pp. 21–35.

[5] L.C.G.J.M. Habets and J.H. van Schuppen, *Control to facet problems for affine systems on simplices and polytopes – With application to control of hybrid systems*, in Proceedings 44th IEEE Conf. on Decision and Control and European Control Conf. 2005, Seville, 2005, pp. 4175–4180.

[6] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen, *Reachability and control synthesis for piecewise-affine hybrid systems on simplices*, IEEE Trans. Aut. Control, 51 (2006), pp. 938–948.

[7] J. Rambau and G.M. Ziegler, *Projections of polytopes and the generalized Baues Conjecture*, Discrete Comput. Geom., 16 (1996), pp. 215–237.

[8] E.D. Sontag, *Nonlinear regulation: the piecewise-linear approach*, IEEE Trans. Aut. Control, 26 (1981), pp. 346–358.

[9] E.D. Sontag, *Interconnected automata and linear systems: a theoretical framework in discrete-time*, in Hybrid Systems III: Verification and Control, R. Alur, T.A. Henzinger and E.D. Sontag, eds., Lecture Notes in Computer Science, Vol. 1066, Springer-Verlag, Berlin, 1996, pp. 436–448.

[10] G.M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, Vol. 152, Springer-Verlag, Berlin, 1995.